An Overview of the Bilinear Hilbert Transform

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A-Exam Presentation
Spring 2018

The Hilbert Transform

- For $f \in \mathcal{S}(\mathbb{R})$, the **Hilbert transform** is given by:

$$Hf(x) := \lim_{\epsilon \to 0} \int_{|t|>\epsilon} \frac{f(x-t)}{t}\, dt.$$

- As a **multiplier operator**, it is:

$$\widehat{Hf}(\xi) = -\pi i \operatorname{sgn}(\xi) \hat{f}(\xi).$$

- The Hilbert transform is an example of a **singular integral operator of Calderón-Zygmund type**. Calderón-Zygmund theory is used to prove the following:

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Theorem

The Hilbert transform is bounded on $L^p(\mathbb{R})$ for every $1 < p < \infty$:

$$||Hf||_{L^p(\mathbb{R})} \leqslant C_p ||f||_{L^p(\mathbb{R})},$$

for some $C_p > 0$.

Calderón-Zygmund Theory Summary

- $L^2 \to L^2$ bounds follow from Plancherel's theorem and the properties of the kernel $1/t$.
- $L^1 \to L^{1,\infty}$ bounds for the Hilbert transform are obtained via the Calderón-Zygmund decomposition.
- $L^p \to L^p$ bounds, for $1 < p < \infty$, follow from interpolating between the above estimates and duality.

BHT and History

The **bilinear Hilbert transform** in the direction $(\alpha, \beta) \in \mathbb{R}^2$ is defined for $f, g \in \mathcal{S}(\mathbb{R})$by

$$BHT_{\alpha,\beta}(f,g)(x) := \lim_{\epsilon \to 0} \int_{|t|>\epsilon} f(x - \alpha t) g(x - \beta t) \frac{dt}{t}.$$

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Non-Uniform Bounds (depending on (α, β))

- *M. Lacey, C. Thiele* '97: $2 < p, q < \infty, 1 < r < 2$
- *M. Lacey, C. Thiele* '99: $1 < p, q \leqslant \infty, 2/3 < r < \infty$ (general case)

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Uniform Bounds (independent of (α, β))

- *C. Thiele* '02: weak estimate in $(2, 2, \infty)$
- *L. Grafakos, X. Li* '04: $2 < p, q < \infty, 1 < r < 2$
- *X. Li* '06: $1 < p, q < 2, 2/3 < r < 1$
- *R. Oberlin, C. Thiele* '11: expected bounds for Walsh model

The Main Theorem

The result that we study in this talk asserts the following:

Theorem (M. Lacey, C. Thiele, 1999)

The bilinear Hilbert transform maps $L^p \times L^q$ *into* L^r *for any* $1 < p, q \leqslant \infty$ *with the property that* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ *and* $2/3 < r < \infty$.

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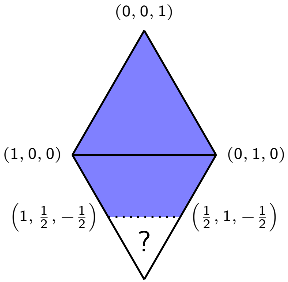

Figure: Range for the BHT operator. The plot contains tuples $\boldsymbol{(1/p, 1/q, 1/r')}$, which in our case must lie on the plane $\boldsymbol{x + y + z = 1}$.

Restricted Weak Type Estimates

Definition

Let $1 < p, q < \infty$ and $0 < r < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. A bilinear operator T is of **restricted weak type (p, q, r)** if for all measurable sets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}$ of finite measure there exists $\mathcal{E}' \subset \mathcal{E}$ with $|\mathcal{E}'| \simeq |\mathcal{E}|$ (called a major subset), such that

$$\left| \int_{\mathbb{R}} T(f_1, f_2)(x) f(x) \; dx \right| \lesssim |\mathcal{E}_1|^{1/p} |\mathcal{E}_2|^{1/q} |\mathcal{E}'|^{1/r'}$$

for every $|f_1| \leqslant \chi_{\mathcal{E}_1}, |f_2| \leqslant \chi_{\mathcal{E}_2}$, and $|f| \leqslant \chi_{\mathcal{E}'}$.

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for every $|f_1| \leqslant \chi_{\mathcal{E}_1}, |f_2| \leqslant \chi_{\mathcal{E}_2}$, and $|f| \leqslant \chi_{\mathcal{E}'}$.

- If T is of restricted weak type (p, q, r), then

$$||T(f_1, f_2)||_{r,\infty} \lesssim ||f_1||_p ||f_2||_q$$

whenever f_1 and f_2 are as above.

Main Theorem Reformulated

The proof of the main theorem can be reduced to proving the following:

Theorem

Fix $\epsilon > 0$, (small). Let $1 < p < 1 + \epsilon$, $2 - \epsilon < q < 2$, and such that for $\frac{1}{r} := \frac{1}{p} + \frac{1}{q}$ one has $2/3 < r < 1$. Then the BHT is of restricted weak type (p, q, r).

Interpolation Details

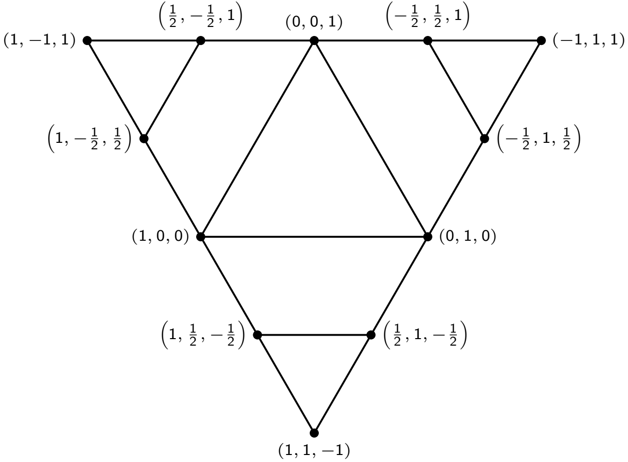

Figure: The three step interpolation to reduce the main theorem to the theorem on the previous slide. The plot contains tuples $(1/p, 1/q, 1/r')$, which in our case must lie on the plane $x + y + z = 1$.

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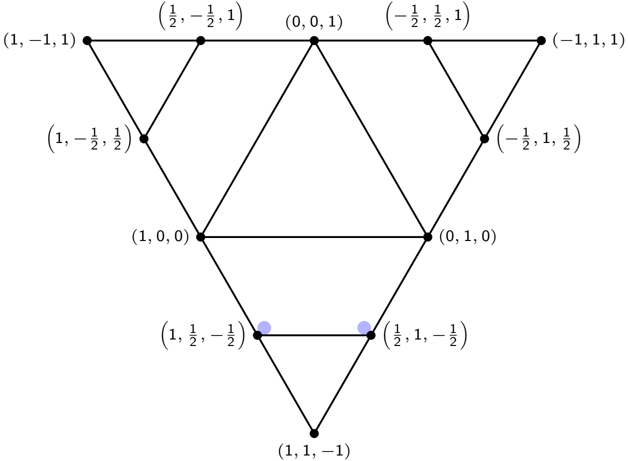

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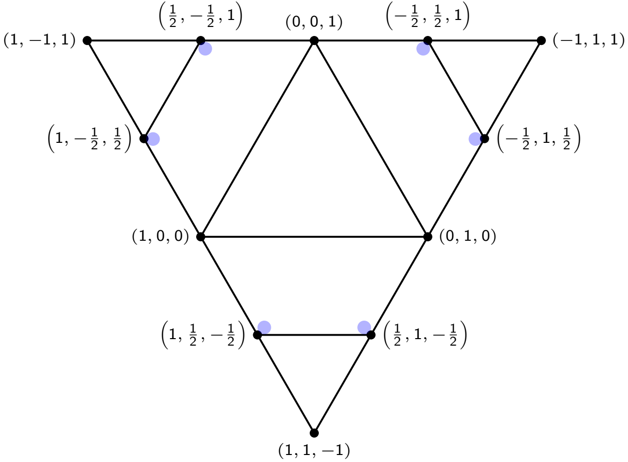

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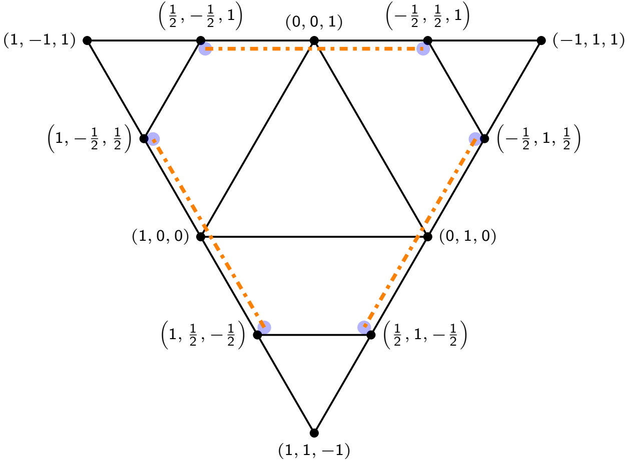

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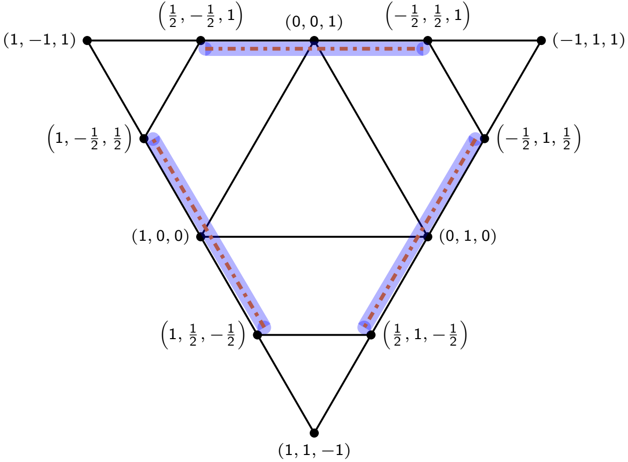

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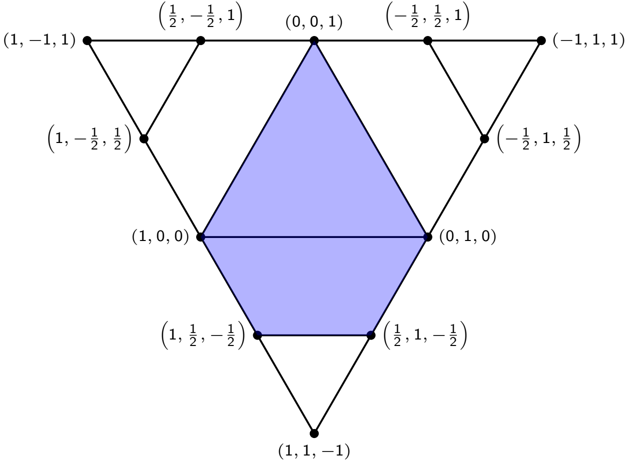

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Comparison with Bilinear Coifmann-Meyer Operators

- A **bilinear Coifmann-Meyer operator** is an operator of type:

$$T(f,g)(x) \mapsto \int_{\mathbb{R}^2} m(\xi_1,\xi_2)\hat{f}(\xi_1)\hat{g}(\xi_2)e^{2\pi i x(\xi_1+\xi_2)}d\xi_1 d\xi_2,$$

where $|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{|\alpha|}}$.

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- The **bilinear Hilbert transform is a bilinear multiplier operator** with a more singular multiplier:

$$BHT(f,g) \to -i\pi \int_{\mathbb{R}^2} \text{sgn}(\xi_1-\xi_2)\hat{f}(\xi_1)\hat{g}(\xi_2)e^{2\pi ix(\xi_1+\xi_2)}d\xi_1 d\xi_2.$$

Comparing Multiplier Singularities

Bilinear Coifmann-Meyer Operator

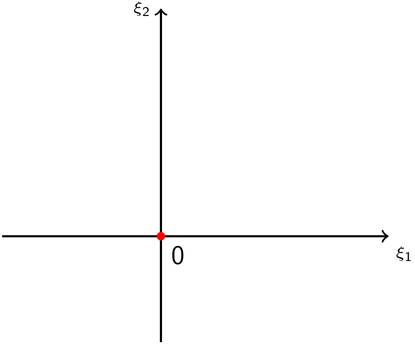

Figure: Singularity point, $(\xi_1, \xi_2) = (0, 0)$, of multiplier

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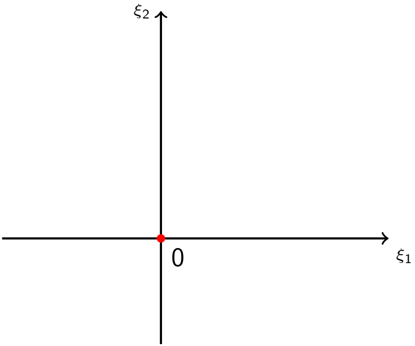

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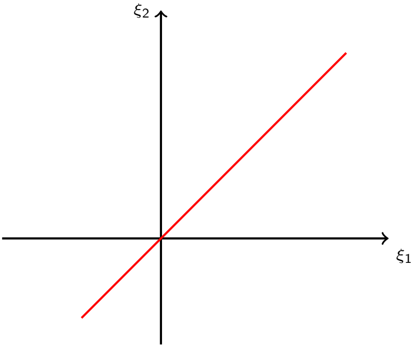

Figure: Singularity line, $\xi_1 = \xi_2$, of BHT multiplier

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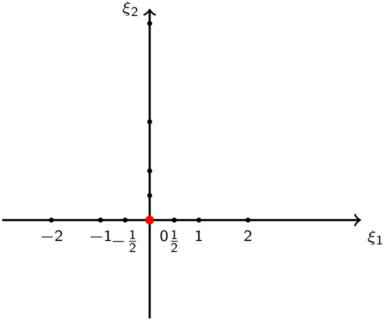

Figure: Whitney rectangles with respect to $(0,0)$

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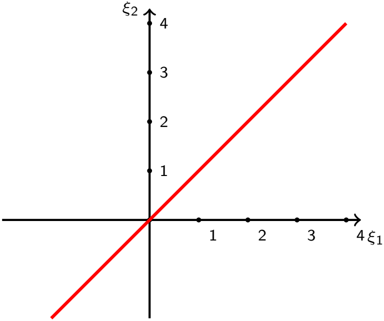

Figure: Whitney squares with respect to $\xi_1 = \xi_2$.

Discrete Representation of the Multiplier

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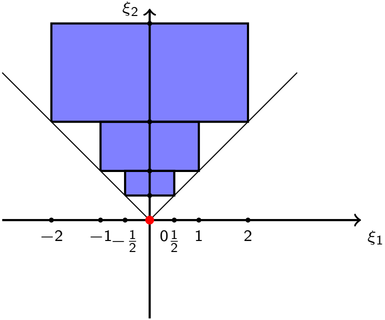

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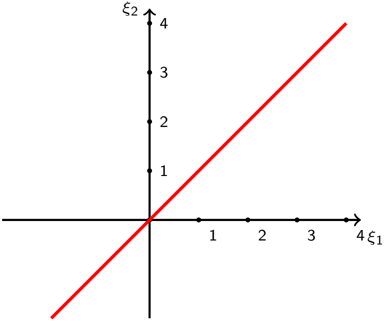

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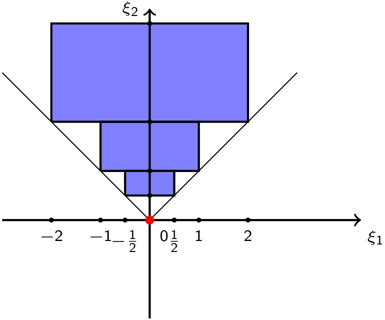

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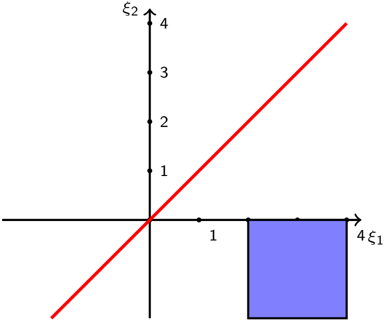

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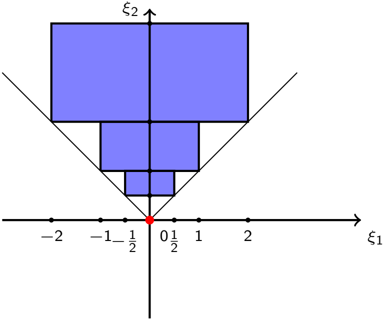

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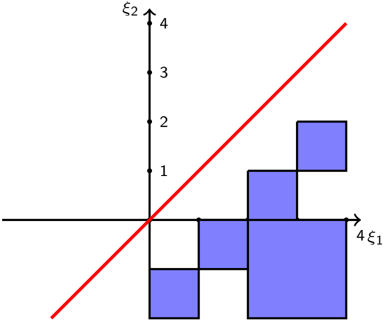

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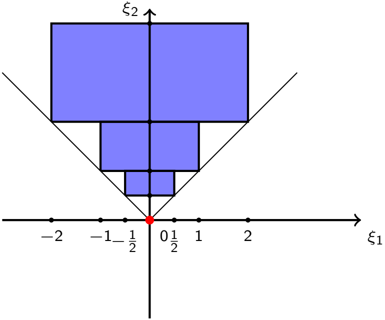

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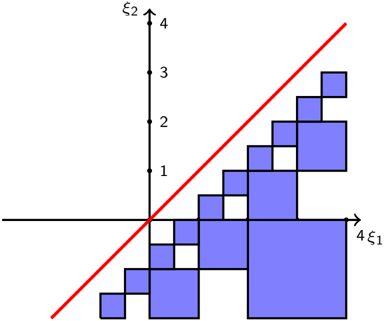

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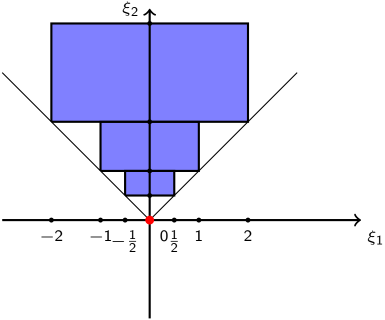

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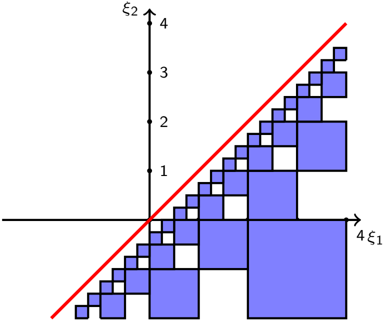

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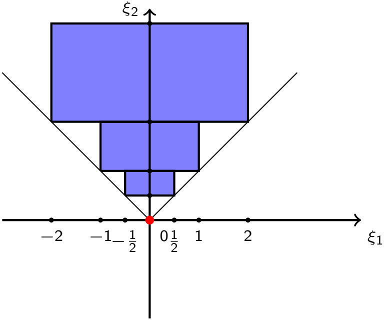

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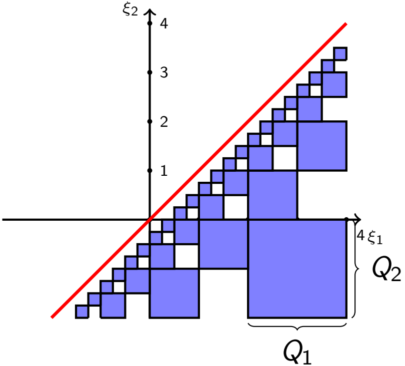

Figure: Whitney squares with respect to $\xi_1 = \xi_2$. Giving model:
$m(\xi_1, \xi_2) = \sum_Q \hat{\phi}_{Q_1}(\xi_1)\hat{\phi}_{Q_2}(\xi_2)\hat{\phi}_{Q_3}(\xi_1 + \xi_2)$

BHT Model Operator

We obtain a **model operator** associated to the BHT given by:

$$BHT_{\mathbb{P}}(f_1, f_2) = \sum_{P \in \mathbb{P}} \frac{1}{|I_P|^{1/2}} \langle f_1, \Phi^1_{P_1} \rangle \langle f_2, \Phi^2_{P_2} \rangle \Phi^3_{P_3}.$$

Each $\boldsymbol{P = (P_1, P_2, P_3)} \in \mathbb{P}$ is a **3-tuple** of tiles in the **phase plane.**

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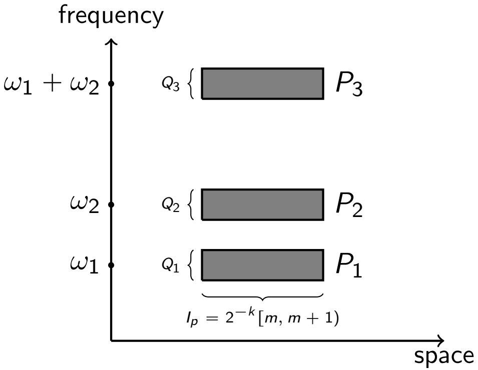

$$P_i = I_p \times Q_i$$

$$|I_p| \cdot |Q_i| = 1$$

Properties of Tri-Tiles

- For every dyadic interval I, there can be a whole column of tri-tiles P s.t. $I_P = I$.
- The position of P_1 or P_2 or P_3 determines position of the rest.
  - Given location of P_1/P_2, then P_2/P_1 lies a number of steps away comparable to C_0.
  - The frequency coordinate of P_3 is essentially a sum of the other two.
- If the frequency intervals of P_1 intersect i.e. all contain ξ_0, then the frequency intervals of the corresponding P_2 tiles are disjoint and **lacunary** away from ξ_0.

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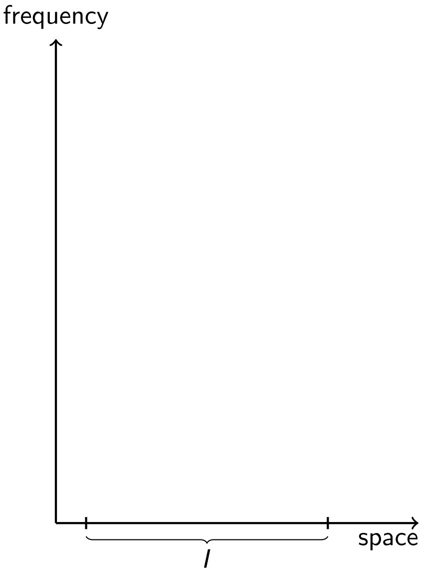

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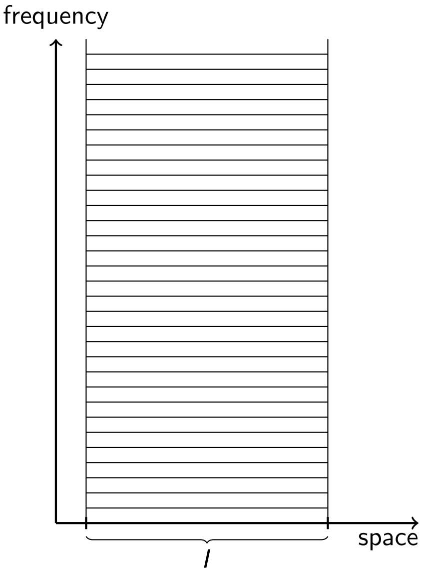

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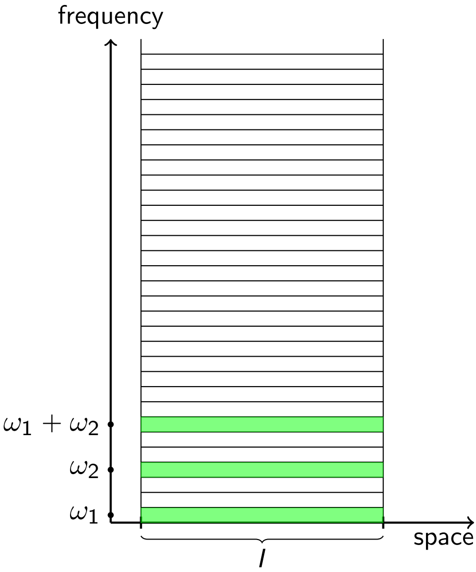

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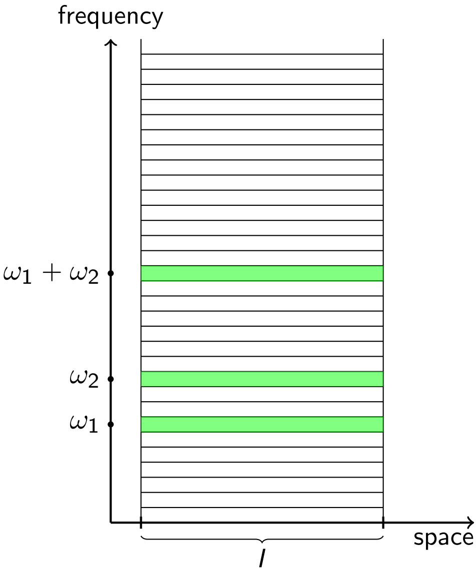

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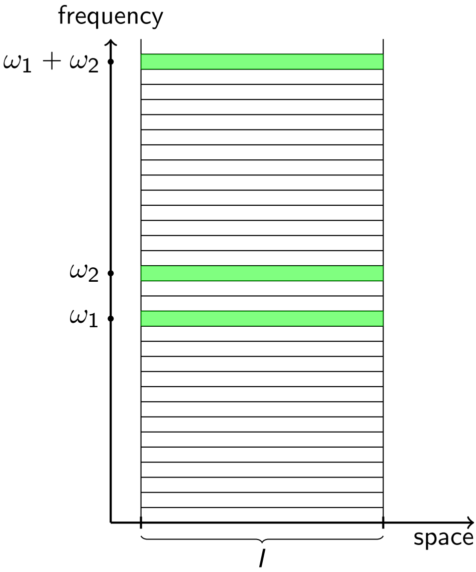

Properties of Tri-Tiles

- For every dyadic interval I, there can be a whole column of tri-tiles P s.t. $I_P = I$.
- The position of P_1 or P_2 or P_3 determines position of the rest.
    - Given location of P_1/P_2, then P_2/P_1 lies a number of steps away comparable to C_0.
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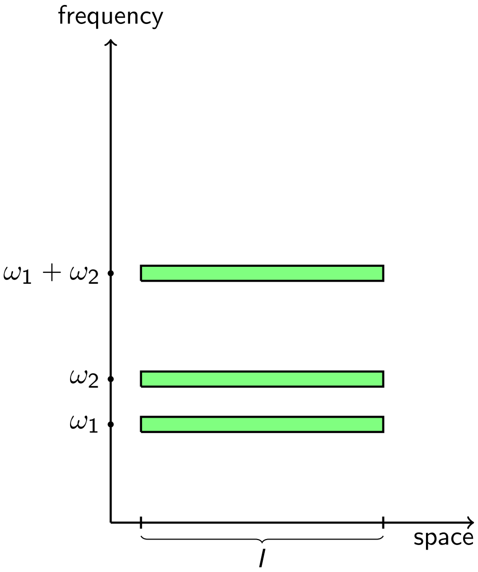

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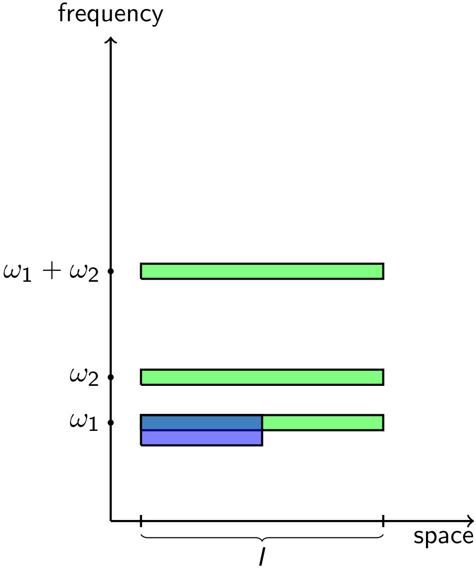

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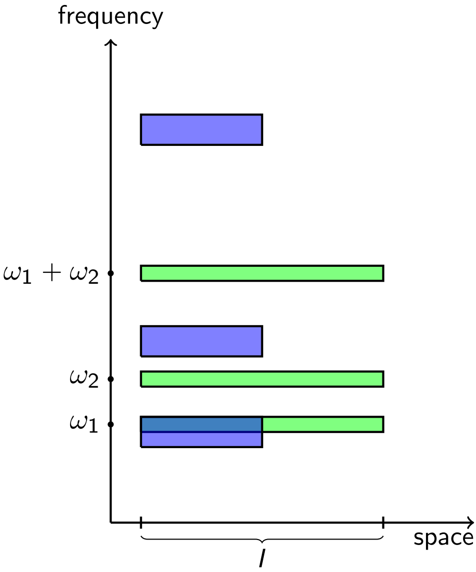

Trees

Definition

Let $\mathbb{P}$ be any collection of tri-tiles. A subcollection $T \subset \mathbb{P}$ is called a **j-tree** provided there exists a tri-tile $\boldsymbol{P_T}$ (called the **top of the tree**) such that

$$I_P \subseteq I_{P_T} \quad \text{and} \quad \omega_{P_T,j} \subseteq 3\omega_{P,j}, \quad \text{for every} \quad P \in T.$$

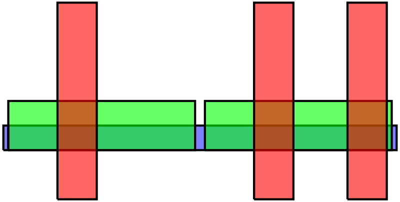

Figure: P_1 tiles in a 1-tree.

L^2 Size

Definition

Let $\mathbb{P}$ be a finite collection of tri-tiles and let $f : \mathbb{R} \to \mathbb{C}$. The **j-size**, for $j \in \{1, 2, 3\}$, of the sequence $\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}}$ is

$$\mathrm{size}_{\mathbb{P}} \left(\langle f, \Phi^j_{P_j} \rangle_P \right) := \sup_{T \subseteq \mathbb{P}} \left(\frac{1}{|I_T|} \sum_{P \in T} |\langle f, \Phi^j_{P_j} \rangle|^2 \right)^{1/2},$$

where T ranges over all trees in $\mathbb{P}$ that are $i-$ trees for $i \neq j$.

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where T ranges over all trees in $\mathbb{P}$ that are $i-$ trees for $i \neq j$.

- Sizes aid in estimating $BHT_{\mathbb{P}}$ when $\mathbb{P}$ consists of a single tree:

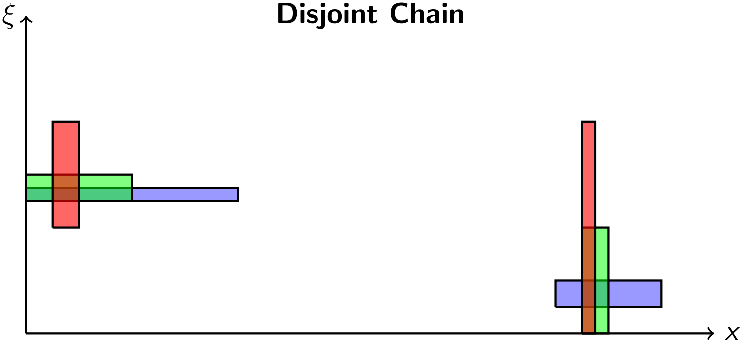
$$\left| \int_{\mathbb{R}} \text{BHT}_T(f_1, f_2)(x) f_3(x) dx \right| \leqslant$$

$$\sum_{P \in T} \frac{1}{|I_P|^{1/2}} |\langle f_1, \Phi^1_{P_1} \rangle| |\langle f_2, \Phi^2_{P_2} \rangle| |\langle f_3, \Phi^3_{P_3} \rangle| \leqslant |I_T| \prod_{j=1}^{3} \text{size} \left(\langle f_j, \Phi^j_{P_j} \rangle_{P \in T} \right)$$

Chain of Strongly i-Disjoint Trees

Let $1 \leqslant i \leqslant 3$. A finite sequence of trees $T_1, \ldots, T_M$ is a **chain of strongly i-disjoint trees** provided that: they are pairwise disjoint and

- If $P \in T_{\ell_1}, P' \in T_{\ell_2} (\ell_1 \neq \ell_2)$ with $2\omega_{P_i} \cap 2\omega_{P'_i} \neq \varnothing$ then $|\omega_{P_i}| \leqslant |\omega_{P'_i}| \implies I_{P'} \cap I_{T_{\ell_1}} = \varnothing$ and $|\omega_{P'_i}| < |\omega_{P_i}| \implies I_P \cap I_{T_{\ell_2}} = \varnothing$.



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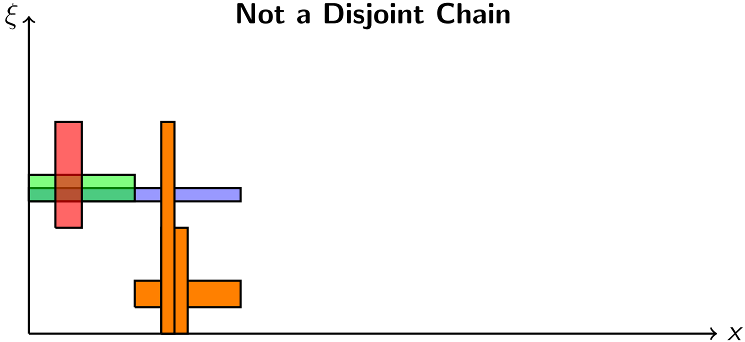
Not a Disjoint Chain



L^2 Energy

Definition

The **j-energy** of the sequence $\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}}$ is

$$\text{energy}\left(\langle f, \Phi^j_{P_j} \rangle_P\right) := \sup_{n \in \mathbb{Z}} \sup_{\mathbb{T}} 2^n \left(\sum_{T \in \mathbb{T}} |I_T| \right)^{1/2},$$

$\mathbb{T}$ ranges over chains of strongly $j-$ disjoint trees (which are i-trees for some $i \neq j$) having the property that

$$\text{size}\left(\langle f, \Phi^j_{P_j} \rangle_P\right) \simeq 2^n.$$

- The energy aids in summing the estimates obtained on each individual tree.

Estimate on the Trilinear Form

The following theorem provides a way of estimating a generic trilinear form associated with $BHT_{\mathbb{P}}(f_1, f_2)$. First we write:

$$\Lambda_{\mathbb{P}}(f_1, f_2, f_3) := \int_{\mathbb{R}} BHT_{\mathbb{P}}(f_1, f_2)(x) f_3(x) dx.$$

Theorem (size-energy estimate)

Let $\mathbb{P}$ be a finite collection of tri-tiles. Then

$$|\Lambda_{\mathbb{P}}(f_1, f_2, f_3)| \lesssim \prod_{j=1}^{3} (\mathrm{size}(\langle f, \Phi^j_{P_j} \rangle_{\mathbb{P}}))^{\theta_j} (\mathrm{energy}(\langle f, \Phi^j_{P_j} \rangle_{\mathbb{P}}))^{1-\theta_j}$$

for any $0 \leqslant \theta_1, \theta_2, \theta_3 < 1$ *with* $\theta_1 + \theta_2 + \theta_3 = 1$.

Stopping-time Decompositions

The following is a key ingredient in the proof of the size-energy duality theorem.

Proposition (stopping-time decomposition)

Let $j \in \{1, 2, 3\}$*. For any* $\mathbb{P}' \subset \mathbb{P}$ *and any* $n \in \mathbb{Z}$ *such that*

$$\text{size}(\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}'}) \leqslant 2^{-n} \text{energy} \left(\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}} \right),$$

One can decompose $\mathbb{P}' = \mathbb{P}^- \cup \mathbb{P}^+$ *in such a way that*

$$\text{size}(\langle f_j, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}^-}) \leqslant 2^{-n-1} \text{energy} \left(\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}} \right)$$

and $\mathbb{P}^+$ *can be written as a disjoint union of trees* $T \in \mathbb{T}$ *such that*

$$\sum_{T \in \mathbb{T}} |I_T| \lesssim 2^{2n}.$$

Proof of Stopping-Time Decomposition

Proof.

(WLOG take j=2) Consider all i-trees T ($i \neq 2$) that are upward $2-$ trees rooted at P_T and satisfy:

$$\left(\frac{1}{|I_T|} \sum_{P \in T} |\langle f, \Phi^j_{P_j} \rangle|^2 \right)^{1/2} > 2^{-n-1} \text{energy} \left(\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}} \right).$$

If there are no such trees, terminate algorithm.

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If there are no such trees, terminate algorithm. Otherwise, choose a maximal T whose center $\xi_{T,i}$ of $\omega_{P_{T,i}}$ is largest. Remove T and $\widetilde{T}$ from $\mathbb{P}'$ and place into $\mathbb{P}^+$ where:

$$\widetilde{T} := \{P \in \mathbb{P}'\backslash T | I_P \subseteq I_{P_T}, \omega_{P_T,2} \subseteq 3\omega_{P,2}\}.$$

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Continue until algorithm terminates. Trees $T_1, T_2, ..., T_M$ form a chain of strongly $2-$ disjoint trees.

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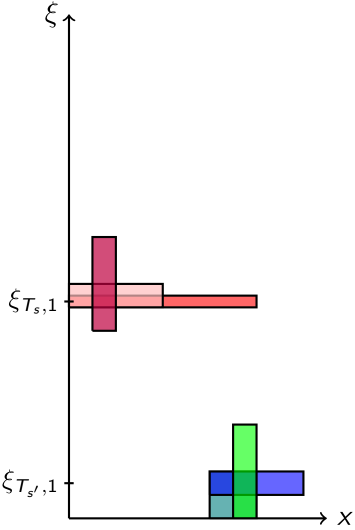
$\square$

Justifying that $T_1, T_2, \ldots, T_M$ form a chain of strongly $j-$disjoint trees

- Assume T_s, $T_{s'}$ do not satisfy the strongly 2-disjointness property.
- So, there are $P \in T_s, P' \in T_{s'}$ with $|\omega_{P_2}| < |\omega_{P_2'}|$ and $I_{P'} \subset I_{T_s}$.
- But $|\omega_{P_2}| < |\omega_{P_2'}|$ implies $\xi_{P_{T_{s'},i}} < \xi_{P_{T_s,i}}$.
- So, T_s is selected before $T_{s'}$.
- Hence, the tri-tile P' was removed in $\widetilde{T}_s$, contradicting that $P' \in T_{s'}$.



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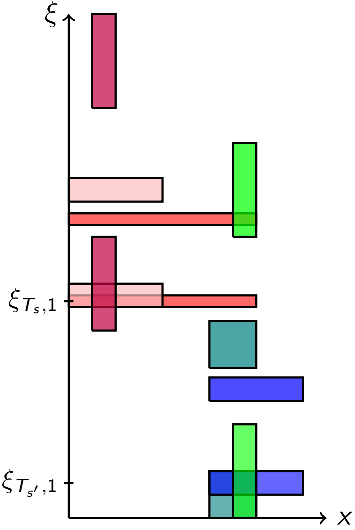

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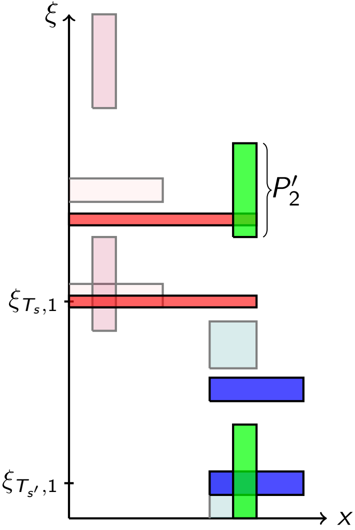

Iterated Stopping-Time Decomposition

$\mathbb{P} - \text{size}(\langle f, \Phi^i_{P_i} \rangle_{P \in \mathbb{P}}) \leq 2^{-k} E_i \quad \text{for} \quad i \in \{1,2,3\}$

$\mathbb{P}^+_k$:

- $\mathbb{P}^+_{1,k}$, $\sum_{T \in \mathbb{T}} |I_T| \lesssim 2^{2k}$
- $\mathbb{P}^-_{1,k+1} - \text{size}(\langle f, \Phi^1_{P_1} \rangle_{P \in \mathbb{P}^-_{1,k+1}}) \leq 2^{-(k+1)} E_1$
  - $\mathbb{P}^+_{2,k}$, $\sum_{T \in \mathbb{T}} |I_T| \lesssim 2^{2k}$
  - $\mathbb{P}^-_{2,k+1} - \text{size}(\langle f, \Phi^2_{P_2} \rangle_{P \in \mathbb{P}^-_{2,k+1}}) \leq 2^{-(k+1)} E_2$
    - $\mathbb{P}^+_{3,k}$, $\sum_{T \in \mathbb{T}} |I_T| \lesssim 2^{2k}$
    - $\mathbb{P}^-_{3,k+1} - \text{size}(\langle f, \Phi^3_{P_3} \rangle_{P \in \mathbb{P}^-_{3,k+1}}) \leq 2^{-(k+1)} E_3$

$\mathbb{P}^+_{k+1}$:

- $\mathbb{P}^+_{1,k+1}$, $\sum_{T \in \mathbb{T}} |I_T| \lesssim 2^{2(k+1)}$
- $\mathbb{P}^-_{1,k+2} - \text{size}(\langle f, \Phi^1_{P_1} \rangle_{P \in \mathbb{P}^-_{1,k+2}}) \leq 2^{-(k+2)} E_1$
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$\ddots$

Iterated Stopping-time Rigorously Stated

Corollary

Let $\mathbb{P}$ be a collection of tri-tiles. One can split $\mathbb{P}$ as

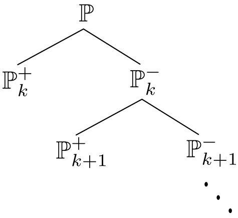
$$\mathbb{P} = \bigcup_{k \in \mathbb{Z}} \mathbb{P}_k, \quad \text{where for} \quad k \in \mathbb{Z} \quad \text{we have}$$

$$\operatorname{size}\left(\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_k} \right) \leqslant \min(2^{-k} E_j, S_j), \quad \text{for every} \quad j = 1, 2, 3.$$

Also, one can cover $\mathbb{P}_k$ by a collection of trees $T \in \mathbb{T}_k$ for which

$$\sum_{T \in \mathbb{T}_k} |I_T| \lesssim 2^{2k}.$$

Iterated Stopping-Time to Deduce Size-Energy Estimate



- $\mathbb{P} = \bigsqcup_{k \in \mathbb{Z}} \mathbb{P}_k^+$
- $\mathbb{P}_k$ covered by trees $T \in \mathbb{T}_k$ for which $\sum_{T \in \mathbb{T}_k} |I_T| \lesssim 2^{2k}$
- $S^j_{\mathbb{P}_k^+} \lesssim 2^{-k} E^j_{\mathbb{P}}$

$$\begin{aligned}
\left| \Lambda_{\mathbb{P}} \left(\frac{f_1}{E_1}, \frac{f_2}{E_2}, \frac{f_3}{E_3} \right) \right| &= \sum_{k \in \mathbb{Z}} \left(\prod_{j=1}^{3} \text{size} \left(\left(\left\langle \frac{f_j}{E_j}, \Phi^j_{P_j} \right\rangle_{P \in \mathbb{P}_k} \right) \right) \right) \sum_{T \in \mathbb{T}_k} |I_T| \\
&\lesssim \sum_{k \in \mathbb{Z}} \left(\prod_{j=1}^{3} \text{size} \left(\left(\left\langle \frac{f_j}{E_j}, \Phi^j_{P_j} \right\rangle_{P \in \mathbb{P}_k} \right) \right) \right) 2^{2k} \\
&\lesssim \left(\frac{S_1}{E_1} \right)^{\theta_1} \left(\frac{S_2}{E_2} \right)^{\theta_2} \left(\frac{S_3}{E_3} \right)^{\theta_3}.
\end{aligned}$$

Bound on the Size

Lemma (Maximal Operator Bound)

Let $j \in \{1, 2, 3\}$, *then for every* $f \in \mathcal{S}(\mathbb{R})$ *one has*

$$\operatorname{size}\left(\langle f, \Phi^j_{P_j}\rangle_{\mathbb{P}}\right) \lesssim \sup_{P \in \mathbb{P}} \frac{1}{|I_P|} \int_{\mathbb{R}} |f(x)| \cdot \widetilde{\chi}_{I_P}(x) dx.$$

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- John-Nirenberg Inequality:

$$\operatorname{size}^j_{\mathbb{P}} := \sup_{T\subset\mathbb{P}} \frac{1}{|I_T|^{1/2}} \left\| \left(\sum_{P\in T} \frac{|\langle f, \phi^j_{P_j}\rangle|^2}{|I_P|} \chi_{I_P} \right)^{1/2} \right\|_2 \simeq$$

$$\sup_{T\subset\mathbb{P}} \frac{1}{|I_T|} \left\| \left(\sum_{P\in T} \frac{|\langle f, \phi^j_{P_j}\rangle|^2}{|I_P|} \chi_{I_P} \right)^{1/2} \right\|_{1,\infty}$$

Bound on the Energy

Lemma (Bessel Inequality)

Let $j \in \{1, 2, 3\}$ *and* $f \in L^2(\mathbb{R})$*. Then*

$$\text{Energy}\left(\langle f, \Phi^j_{P_j} \rangle_{\mathbb{P}}\right) \lesssim ||f||_2.$$

Lemma (Bessel Inequality)

Let $j \in \{1,2,3\}$ *and* $f \in L^2(\mathbb{R})$. *Then*

$$\text{Energy}\left(\langle f, \Phi^j_{P_j}\rangle_{\mathbb{P}}\right) \lesssim ||f||_2.$$

- One invokes **strong j-disjointness** of trees $T \in \mathbb{T}$ and **almost orthogonality** of the corresponding wave packets $\{\Phi^j_{P_j}\}_{P\in T}$:

$$E\left(\langle f, \Phi^j_{P_j}\rangle_P\right)^2 = 2^n\left(\sum_{T\in\mathbb{T}} |I_T|\right) \lesssim 2^n 2^{-n}\left(\sum_{T\in\mathbb{T}}\left(\sum_{P\in T} |\langle f, \Phi^j_{P_j}\rangle|^2\right)\right)$$

$$= \left|\left\langle \sum_T \sum_{P\in T} \langle f, \Phi_{P_j}\rangle \Phi_{P_j}, f\right\rangle\right| \lesssim ||f||_2 \left\|\sum_T \sum_{P\in T} \langle f, \Phi_{P_j}\rangle \Phi_{P_j}\right\|_2 .$$

Proof of Main Theorem

- Fix measurable sets $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}$ of finite measure. **Our goal is to construct a subset $\mathcal{E}' \subset \mathcal{E}$** with $|\mathcal{E}'| \simeq |\mathcal{E}|$ and such that

$$\left| \sum_{P \in \mathbb{P}} \frac{1}{|I_P|^{1/2}} \langle f_1, \Phi^1_{P_1} \rangle \langle f_2, \Phi^2_{P_2} \rangle \langle f_3, \Phi^3_{P_3} \rangle \right| \lesssim |\mathcal{E}_1|^{1/p} |\mathcal{E}_2|^{1/q} |\mathcal{E}'|^{1/r'} \tag{1}$$

For every $|f_1| \leqslant \chi_{\mathcal{E}_1}, |f_2| \leqslant \chi_{\mathcal{E}_2}$, and $|f| \leqslant \chi_{\mathcal{E}'}$.

Proof of Main Theorem (Continued)

- Define first an **exceptional set**

$$\Omega := \{x : Mf_1(x) > C|\mathcal{E}_1|\} \bigcup \{x : Mf_2(x) > C|\mathcal{E}_2|\},$$

where M is the usual Hardy-Littlewood maximal operator.

Proof of Main Theorem (Continued)

- Define first an **exceptional set**

$$\Omega := \{x : Mf_1(x) > C|\mathcal{E}_1|\} \bigcup \{x : Mf_2(x) > C|\mathcal{E}_2|\},$$

where M is the usual Hardy-Littlewood maximal operator.

- Set $\boldsymbol{\mathcal{E}' := \mathcal{E} \backslash \Omega}$. It satisfies $|\mathcal{E}'| \simeq |\mathcal{E}|$ if C is sufficiently large enough.

Proof of Main Theorem (Continued)

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$$\Omega := \{x : Mf_1(x) > C|\mathcal{E}_1|\} \bigcup \{x : Mf_2(x) > C|\mathcal{E}_2|\},$$

  where M is the usual Hardy-Littlewood maximal operator.

- Set $\boldsymbol{\mathcal{E}' := \mathcal{E} \backslash \Omega}$. It satisfies $|\mathcal{E}'| \simeq |\mathcal{E}|$ if C is sufficiently large enough.

- To be able to estimate we split our collection of tri-tiles $\mathbb{P}$ as follows:

$$\mathbb{P} = \bigcup_{d \geqslant 0} \mathbb{P}_d,$$

  where $\mathbb{P}_d$ contains all the tri-tiles in $\mathbb{P}$ having the property that

$$2^d \leqslant 1 + \frac{\text{dist}(I_p, \Omega^c)}{|I_P|} < 2^{d+1}.$$

Proof of Main Theorem (Continued)

So, one has

$$|\Lambda_{\mathbb{P}}(f_1, f_2, f_3)| \leqslant \sum_{d=0}^{\infty} |\Lambda_{\mathbb{P}_d}(f_1, f_2, f_3)|$$

$$\lesssim \sum_{d=0}^{\infty} \left(\prod_{j=1}^{3} \left(\text{size} \left(\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \right)^{\theta_j} \left(\text{energy} \left(\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \right)^{1-\theta_j} \right)$$

for any $0 \leqslant \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

Proof of Main Theorem (Continued)

So, one has

$$|\Lambda_{\mathbb{P}}(f_1, f_2, f_3)| \leqslant \sum_{d=0}^{\infty} |\Lambda_{\mathbb{P}_d}(f_1, f_2, f_3)|$$

$$\lesssim \sum_{d=0}^{\infty} \left(\prod_{j=1}^{3} \left(\text{size}\left(\langle f, \Phi^j_{P_j}\rangle_{P\in\mathbb{P}_d}\right)\right)^{\theta_j} \left(\text{energy}\left(\langle f, \Phi^j_{P_j}\rangle_{P\in\mathbb{P}_d}\right)\right)^{1-\theta_j} \right)$$

for any $0 \leqslant \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

- **Size Lemma (Maximal Function Bound):**

$$\text{size}\left(\langle f_i, \Phi^j_{P_j}\rangle_{P\in\mathbb{P}_d}\right) \lesssim \sup_{P\in\mathbb{P}_d} \frac{1}{|I_P|} \int f_i \widetilde{\chi}^M_{I_P} dx \implies \begin{cases} S_i \lesssim 2^d |\mathcal{E}_i| & i=1,2 \\ S_3 \lesssim 2^{-Md} |\mathcal{E}'| \end{cases}$$

Proof of Main Theorem (Continued)

So, one has

$$|\Lambda_{\mathbb{P}}(f_1, f_2, f_3)| \leqslant \sum_{d=0}^{\infty} |\Lambda_{\mathbb{P}_d}(f_1, f_2, f_3)|$$

$$\lesssim \sum_{d=0}^{\infty} \left(\prod_{j=1}^{3} \left(\text{size}\left(\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \right)^{\theta_j} \left(\text{energy}\left(\langle f, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \right)^{1-\theta_j} \right)$$

for any $0 \leqslant \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$.

- **Size Lemma (Maximal Function Bound):**

$$\text{size}\left(\langle f_i, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \lesssim \sup_{P \in \mathbb{P}_d} \frac{1}{|I_P|} \int f_i \widetilde{\chi}^M_{I_P} dx \implies \begin{cases} S_i \lesssim 2^d |\mathcal{E}_i| & i=1,2 \\ S_3 \lesssim 2^{-Md} |\mathcal{E}'| \end{cases}$$

- **Energy Lemma (Bessel Inequality):**

$$\text{energy}\left(\langle f_i, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \lesssim ||f_i||_2 \implies \text{energy}\left(\langle f_i, \Phi^j_{P_j} \rangle_{P \in \mathbb{P}_d} \right) \lesssim |\mathcal{E}_i|^{1/2}.$$

Proof of Main Theorem (Continued)

So,

$$\lesssim \sum_{d=0}^{\infty} \left(\prod_{j=1}^{3} \left(\text{size}\left(\langle f, \Phi_{P_j}^{j} \rangle_{P \in \mathbb{P}_d} \right) \right)^{\theta_j} \left(\text{energy}\left(\langle f, \Phi_{P_j}^{j} \rangle_{P \in \mathbb{P}_d} \right) \right)^{1-\theta_j} \right)$$

$$\lesssim \sum_{d=0}^{\infty} \left(2^d |\mathcal{E}_1| \right)^{\theta_1} |\mathcal{E}_1|^{(1-\theta_1)/2} \left(2^d |\mathcal{E}_2| \right)^{\theta_2} |\mathcal{E}_2|^{(1-\theta_2)/2} 2^{-Md\theta_3}$$

$$= \sum_{d=0}^{\infty} 2^{d(\theta_1+\theta_2-M\theta_3)} |\mathcal{E}_1|^{(1+\theta_1)/2} |\mathcal{E}_2|^{(1+\theta_2)/2} \lesssim |\mathcal{E}_1|^{(1+\theta_1)/2} |\mathcal{E}_2|^{(1+\theta_2)/2}$$

- Setting $1/p \coloneqq (1+\theta_1)/2$ and $1/q = (1+\theta_2)/2$ completes the proof.

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Thank you for your attention!